Generalized β-conformal change of Finsler metrics*

Nabil L. Youssef†, S. H. Abed‡ and S. G. Elgendi♯

†Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt
nlyoussef2003@yahoo.fr, nyoussef@frcu.eun.eg

‡Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt
sabed52@yahoo.fr, sabed@frcu.eun.eg

♯Department of Mathematics, Faculty of Science,
Benha University, Benha, Egypt
salahelgendi@yahoo.com

Abstract. In this paper, we introduce and investigate a general transformation or change of Finsler metrics, which is referred to as a generalized β-conformal change:

$$L(x,y) \longrightarrow \overline{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y)).$$

This transformation combines both β-change and conformal change in a general setting. The change, under this transformation, of the fundamental Finsler connections, together with their associated geometric objects, are obtained. Some invariants and various special Finsler spaces are investigated under this change. The most important changes of Finsler metrics existing in the literature are deduced from the generalized β-conformal change as special cases.

Keywords: Generalized β-conformal change, β-conformal change, β- change, conformal change, Randers change, Berwald space, Landesberg space, Locally Minkowskian space.

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Introduction

In the context of Riemannian geometry, there is not only a complete local theory, but also a complete global theory, with many practitioners working on both approaches. In this sense, Riemannian geometry is indeed a complete theory. However, the situation in Finsler geometry is substantially different. Finsler geometry was first introduced locally by Finsler himself, to be studied by many eminent mathematicians for its theoretical importance and applications in the variational calculus, mechanics and theoretical physics. Moreover, the dependence of the fundamental function $L(x, y)$ on both the positional argument x and directional argument y offers the possibility to use it to describe the **anisotropic** properties of the physical space.

Let $F^n = (M, L)$ be an n-dimensional Finsler manifold. For a differential one-form $\beta(x, dx) = b_i(x)dx^i$ on M, G. Randers [12], in 1941, introduced a special Finsler space defined by the change $\overline{L} = L + \beta$, where L is Riemannian, to consider a unified field theory.[1] M. Masumoto [10], in 1974, studied Randers space and generalized Randers space in which L is Finslerian. V. Kropina [8] introduced the change $\overline{L} = L^2/\beta$, where L is Reimannian, which has been studied by many authors such as Shibata [13] and Matsumoto [9]. Randers and Kropina spaces are closely related to physics and so Finsler spaces with these metrics have been studied by many authors, from various standpoint in the physical and mathematical aspects ([3], [4], [11], [15], [16], [17]). It was also applied to the theory of the electron microscope by R. S. Ingarden [6]. For a Kropina space (the Finsler space equipped with Kropinas metric), there are close relations between the Kropina metric and the Lagrangian function of analytic dynamics [13]. In 1984, C. Shibata [14] studied the general case of any β-change, that is, $\overline{L} = f(L, \beta)$ which generalizes many changes in Finsler geometry ([8], [10], [19]). In this context, he investigated the change of torsion and curvature tensors corresponding to the above transformation. In addition, he also studied some special Finsler spaces corresponding to specific forms of the function $f(L, \beta)$.

On the other hand, in 1976, M. Hashiguchi [5] studied the conformal change of Finsler metrics, namely, $\overline{L} = e^{\sigma(x)}L$. In particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by many authors ([7], [18]). In 2008, S. Abed ([1], [2]) introduced the transformation $\overline{L} = e^{\sigma(x)}L + \beta$, thus generalizing the conformal, Randers and generalized Randers changes. Moreover, he established the relationships between some important tensors associated with (M, L) and the corresponding tensors associated with $(M, \overline{L})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M, L) and the transformed Cartan connection associated with $(M, \overline{L})$.

In this paper, we deal with a general change of Finsler metrics defined by:

$$L(x, y) \longrightarrow \overline{L}(x, y) = f(e^{\sigma(x)}L(x, y), \beta(x, y)) = f(\widetilde{L}, \beta),$$

[1] In 1941, G. Randers published his paper "On an asymmetrical metric in the four-space of general relativity". In this paper, Randers considered the simplest possible asymmetrical generalization of a Riemannian metric. Adding a 1-form to the existing Riemannian struture, he was the first to introduce a special Finsler space. This space - which became known in the literature as a Randers space - proved to be mathematically and physically very important. It was one of the first attempts to study a physical theory in the wider context of Finsler geometry, although Randers was not aware that the geometry he used was a special type of Finsler geometry.

where f is a positively homogeneous function of degree one in $\widetilde{L} := e^{\sigma}L$ and β. This change will be referred to as a generalized β-conformal change. It is clear that this change is a generalization of the above mentioned changes and deals simultaneously with β-change and conformal change. It combines also the special case of Shibata ($\overline{L} = f(L, \beta)$) and that of Abed ($\overline{L} = e^{\sigma}L + \beta$).

The present paper is organized as follows. In section 1, the relationship between the Cartan connection associated with (M, L) and the transformed Cartan connection associated with $(M, \overline{L})$ is obtained (Theorem 1.12). The properties that σ being homothetic, b_i being Cartan-parallel and the difference tensor being zero are investigated (Theorems 1.14, 1.15 and 1.16). The coefficients of the fundamental linear connections of Finsler geometry are computed (Theorem 1.18).

In section 2, the torsion and curvature tensor fields of the fundamental linear connections, corresponding to a generalized β-conformal change, are obtained (Theorem 2.1). Some invariants are found (Corollary 2.2) and some properties concerning certain special Finsler spaces are investigated (Theorems 2.3, 2.5 and 2.6).

Finally, in section 3, many interesting changes of Finsler metrics are obtained as special cases form the present change.

Notations

Throughout the present paper, (M, L) denotes an n-dimensional C^{∞} Finsler manifold; L being the fundamental Finsler function. Let (x^i) be the coordinates of any point of the base manifold M and (y^i) a supporting element at the same point. We use the following notations:

∂_i: partial differentiation with respect to x^i,

$\dot{\partial}_i$: partial differentiation with respect to y^i (basis vector fields of the vertical bundle),

$g_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = \dot{\partial}_i\dot{\partial}_j E$: the Finsler metric tensor; $E := \frac{1}{2}L^2$: the energy function,

$l_i := \dot{\partial}_i L = g_{ij}l^j = g_{ij}\frac{y^j}{L}$: the normalized supporting element; $l^i := \frac{y^i}{L}$,

$l_{ij} := \dot{\partial}_i l_j$,

$h_{ij} := L l_{ij} = g_{ij} - l_i l_j$: the angular metric tensor,

$C_{ijk} := \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2$: the Cartan tensor,

G^i: the components of the canonical spray associated with (M, L),

$N^i_j := \dot{\partial}_j G^i$: the Barthel or Cartan nonlinear connection associated with (M, L),

$G^i_{jh} := \dot{\partial}_h N^i_j = \dot{\partial}_h\dot{\partial}_j G^i$,

$\delta_i := \partial_i - N^r_i\dot{\partial}_r$: the basis vector fields of the horizontal bundle,

$C^i_{jk} := g^{ri}C_{rjk} = \frac{1}{2}g^{ir}\dot{\partial}_k g_{rj}$: the h(hv)-torsion tensor,

$\gamma^i_{jk} := \frac{1}{2}g^{ir}(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk})$: the Christoffel symbols with respect to ∂_i,

$\Gamma^i_{jk} := \frac{1}{2}g^{ir}(\delta_j g_{kr} + \delta_k g_{jr} - \delta_r g_{jk})$: the Christoffel symbols with respect to δ_i,

$\quad = \gamma^i_{jk} + g^{it}(C_{jkr}N^r_t - C_{tkr}N^r_j - C_{jtr}N^r_k)$.

We have:

- The canonical spray G: $G^h = \frac{1}{2}\gamma^h_{ij}y^i y^j$.

- The Barthel connection N: $N^i_j = \dot{\partial}_j G^i = G^i_{jh}y^h = \Gamma^i_{jh}y^h$.

- The Cartan connection $C\Gamma$: $(\Gamma^i_{jk}, N^i_j, C^i_{jk})$.

- The Chern (Rund) connection $R\Gamma$: $(\Gamma^i_{jk}, N^i_j, 0)$.

- The Hashiguchi connection $H\Gamma$: $(G^i_{jk}, N^i_j, C^i_{jk})$.

- The Berwald connection $B\Gamma$: $(G^i_{jk}, N^i_j, 0)$.

For a Finsler connection $(\Gamma^i_{jk}, N^i_j, C^i_{jk})$, we define:
$X^i_{j|k} := \delta_k X^i_j + X^r_j \Gamma^i_{rk} - X^i_r \Gamma^r_{jk}$: the horizontal covariant derivative of X^i_j,
$X^i_j|_k := \dot{\partial}_k X^i_j + X^r_j C^i_{rk} - X^i_r C^r_{jk}$: the vertical covariant derivative of X^i_j.

Transvecting with y^j will be denoted by the subscript 0 (excluding p_0, q_0, s_0). For example, we write B^i_{j0} for $B^i_{jk} y^k$.

Finally, the following special symbols will also be used:

- $\Theta_{(j,k,r)}\{A_{jkr}\} := A_{jkr} - A_{krj} - A_{rjk}$.

- $\mathfrak{A}_{(j,k)}\{A_{jk}\} := A_{jk} - A_{kj}$: the alternative sum with respect to the indices j and k.

1. Changes of connections

Let $F^n = (M, L)$ be an n-dimensional C^∞ Finsler manifold with fundamental function $L = L(x, y)$. Consider the following change of Finsler structures which will be referred to as a generalized β-conformal change:

$$L(x, y) \longrightarrow \overline{L}(x, y) = f(e^{\sigma(x)} L(x, y), \beta(x, y)), \tag{1.1}$$

where f is a positively homogeneous function of degree one in $e^\sigma L$ and β and $\beta = b_i(x) dx^i$. Assume that $\overline{F}^n = (M, \overline{L})$ has the structure of a Finsler space. Entities related to $\overline{F}^n$ will be denoted by barred symbols.

We define

$$f_1 := \frac{\partial f}{\partial \widetilde{L}}, \qquad f_2 := \frac{\partial f}{\partial \beta}, \qquad f_{12} := \frac{\partial^2 f}{\partial \widetilde{L} \partial \beta}, \cdots, etc.,$$

where $\widetilde{L} = e^\sigma L$. We use the following notations:

$$\begin{aligned} q &:= f f_2, & p &:= f f_1 / L, \\ q_0 &:= f f_{22}, & p_0 &:= f_2^2 + q_0, \\ q_{-1} &:= f f_{12} / L, & p_{-1} &:= q_{-1} + p f_2 / f, \\ q_{-2} &:= f(e^\sigma f_{11} - f_1/L)/L^2, & p_{-2} &:= q_{-2} + e^\sigma p^2 / f^2. \end{aligned}$$

Note that the subscript under the the above geometric objects indicates the degree of homogeneity of these objects. We also use the notations:

$$b^i = g^{ij} b_j, \quad m_i := b_i - (\beta/L^2) y_i \neq 0, \quad \sigma_i := \partial_i \sigma, \quad p_{02} := \frac{\partial p_0}{\partial \beta}.$$

The following lemmas enable us to compute the geometric objects associated with the space $\overline{F}^n$ obtained from F^n by a generalized β-conformal change. They can be proved by making use of Euler theorem of homogenous functions and the homogeneity properties of p, p_0, p_{-1}, p_{-2}; q, q_0, q_{-1}, q_{-2}.

Lemma 1.1. *The following identities hold:*

(a) $e^\sigma L f_1 + \beta f_2 = f$,

(b) $e^\sigma L f_{12} + \beta f_{22} = 0$,

(c) $e^\sigma L f_{11} + \beta f_{12} = 0$.

Lemma 1.2. *The following identities hold:*

(a) $q_0 \beta + e^\sigma q_{-1} L^2 = 0$,

(b) $q_{-1}\beta + q_{-2} L^2 = -p$,

(c) $p_0 \beta + e^\sigma p_{-1} L^2 = q$,

(d) $p_{-1}\beta + p_{-2} L^2 = 0$,

(e) $q\beta + e^\sigma p L^2 = f^2$.

Lemma 1.3. *The following identities hold:*

(a) $\dot{\partial}_i q = p_0 m_i + q/L l_i$,

(b) $\dot{\partial}_i p = p_{-1} m_i$,

(c) $\dot{\partial}_i p_0 = p_{02} m_i$,

(d) $\dot{\partial}_i p_{-1} = -e^{-\sigma}(\beta/L^2) p_{02} m_i - (p_{-1}/L) l_i$,

(e) $\dot{\partial}_i p_{-2} = [e^{-\sigma}(\beta^2/L^4) p_{02} - (p_{-1}/L^2)] m_i + p_{-1}(2\beta/L^3) l_i$.

Lemma 1.4. *The following identities hold:*

(a) $\partial_k q = p_0 N^r_k m_r + q N^r_k l_r/L + p_0 b_{0|k} + e^\sigma L^2 p_{-1}\sigma_k$,

(b) $\partial_k p = p_{-1} N^r_k m_r + p_{-1} b_{0|k} + (p - \beta p_{-1})\sigma_k$,

(c) $\partial_k p_0 = p_{02}(N^r_k m_r + b_{0|k} - \beta\sigma_k)$,

(d) $\partial_k p_{-1} = -(p_{-1}/L) N^r_k l_r - e^{-\sigma}(\beta/L^2)(p_{02} N^r_k m_r + p_{02} b_{0|k}) + e^{-\sigma}(\beta^2/L^2) p_{02}\sigma_k$,

(e) $$\begin{aligned}\partial_k p_{-2} = {} & [e^{-\sigma}(\beta^2/L^4) p_{02} - (p_{-1}/L^2)] N^r_k m_r + (2\beta p_{-1}/L^3) N^r_k l_r \\ & + [e^{-\sigma}(\beta^2/L^4) p_{02} - (p_{-1}/L^2)] b_{0|k} - e^{-\sigma}(\beta^3/L^4) p_{02}\,\sigma_k.\end{aligned}$$

Now, using Lemma 1.1, we get

Proposition 1.5. *Under a generalized β-conformal change, we have:*

(a) $\bar{l}_i = e^\sigma f_1 l_i + f_2 b_i$,

(b) $\overline{h}_{ij} = e^\sigma p\, h_{ij} + q_0 m_i m_j$,

(c) $\overline{g}_{ij} = e^{\sigma} p\, g_{ij} + p_0\, b_i b_j + e^{\sigma} p_{-1}(b_i y_j + b_j y_i) + e^{\sigma} p_{-2}\, y_i\, y_j.$

Proof. As an illustration, we prove **(c)** only.

$$\begin{aligned}
\overline{g}_{ij} &= \dot{\partial}_j \dot{\partial}_i (\frac{1}{2}\overline{L}^2) = \dot{\partial}_i (f(e^{\sigma} f_1 l_j + f_2 b_j)) \\
&= (e^{\sigma} f_1 l_i + f_2 b_i)(e^{\sigma} f_1 l_j + f_2 b_j) + f(e^{\sigma} f_1 l_{ij} + e^{\sigma}(e^{\sigma} f_{11} l_i + f_{12} b_i) l_j) \\
&\quad + f(e^{\sigma} f_{12} l_i + f_{22} b_i) b_j \\
&= e^{\sigma} p\, g_{ij} + e^{\sigma}[e^{\sigma}(f_1^2/L^2) + e^{\sigma}(f f_{11}/L^2) - (f f_1/L^3)] y_i y_j + (f^2 + q_0) b_i b_j \\
&\quad + e^{\sigma}[(f_1 f_2/L) + (f f_{12}/L)(b_i y_j + b_j y_i)] \\
&= e^{\sigma} p\, g_{ij} + p_0 b_i b_j + e^{\sigma} p_{-1}(b_i y_j + b_j y_i) + e^{\sigma} p_{-2} y_i y_j.
\end{aligned}$$

$\square$

Lemma 1.2 helps us to compute the inverse metric $\overline{g}^{ij}$ of the metric $\overline{g}_{ij}$.

Proposition 1.6. *Under a generalized β-conformal change, the inverse metric $\overline{g}^{ij}$ of the metric $\overline{g}_{ij}$ is given by:*

$$\overline{g}^{ij} = (e^{-\sigma}/p) g^{ij} - s_0 b^i b^j - s_{-1}(y^i b^j + y^j b^i) - s_{-2} y^i y^j,$$

where

$s_0 := e^{-\sigma} f^2 q_0/(\varepsilon p L^2),\ \ s_{-1} := p_{-1} f^2/(p\,\varepsilon L^2),\ \ s_{-2} := p_{-1}(e^{\sigma} m^2 p L^2 - b^2 f^2)/(\varepsilon p \beta L^2),$

$\varepsilon := f^2(e^{\sigma} p + m^2 q_0)/L^2 \neq 0, \quad m^2 = g^{ij} m_i m_j = m^i m_i.$

Remark 1.7. *The quantities s_0, s_{-1} and s_{-2} satisfy:*

$$\beta s_0 + L^2 s_{-1} = q/\varepsilon,$$

$$b^2 s_{-1} + \beta s_{-2} = e^{\sigma} p_{-1} m^2/\varepsilon.$$

Proposition 1.8. *Under a generalized β-conformal change, we have:*

(a) *The Cartan tensor $\overline{C}_{ijk}$ is expressed in terms of C_{ijk} as*

$$\overline{C}_{ijk} = e^{\sigma} p\, C_{ijk} + V_{ijk}, \tag{1.2}$$

(b) *The (h)hv-torsion tensor $\overline{C}^l_{ij}$ is expressed in terms of C^l_{ij} as*

$$\overline{C}^l_{ij} = C^l_{ij} + M^l_{ij}\,, \tag{1.3}$$

where

$$V_{ijk} := \frac{e^{\sigma} p_{-1}}{2}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{p_{02}}{2} m_i m_j m_k$$

and

$$\begin{aligned}
M^l_{ij} &:= \frac{1}{2p}[e^{-\sigma} m^l - p\, m^2(s_0 b^l + s_{-1} y^l)](e^{\sigma} p_{-1} h_{ij} + p_{02} m_i m_j) \\
&\quad - e^{\sigma}(s_0 b^l + s_{-1} y^l)(p\, C_{isj} b^s + p_{-1} m_i m_j) \\
&\quad + \frac{p_{-1}}{2p}(h^l_i m_j + h^l_j m_i); \\
h^i_j &= g^{il} h_{lj}.
\end{aligned}$$

Proof. We prove **(a)** only. Differentiating $\overline{g}_{ij}$ (Proposition 1.5) with respect to y^k, using Lemma 1.3, we have

$$\begin{aligned}
2\overline{C}_{ijk} &= 2e^{\sigma}p\,C_{ijk} + e^{\sigma}g_{ij}p_{-1}m_k + p_{02}b_ib_jm_k + e^{\sigma}p_{-1}(b_ig_{jk} + b_jg_{ik}) \\
&\quad -e^{\sigma}(b_iy_j + b_jy_i).[e^{-\sigma}(\beta/L^2)p_{02}m_k + (p_{-1}/L)l_k] + e^{\sigma}p_{-2}g_{ik}y_j \\
&\quad +e^{\sigma}p_{-2}y_ig_{jk} + e^{\sigma}y_iy_j[(e^{-\sigma}(\beta^2/L^4)p_{02} - (p_{-1}/L^2))m_k + (2\beta/L^3)l_k] \\
&= 2e^{\sigma}p\,C_{ijk} + e^{\sigma}p_{-1}(h_{ij}m_k + h_{jk}b_i + h_{ik}b_j) + p_{02}m_im_jm_k \\
&\quad -e^{\sigma}p_{-1}(\beta/L)(h_{ik}l_j + h_{jk}l_i + 2l_il_jl_k) + 2e^{\sigma}(\beta/L)p_{-1}l_il_jl_k \\
&= 2e^{\sigma}p\,C_{ijk} + e^{\sigma}p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + p_{02}m_im_jm_k.
\end{aligned}$$

□

The transformed Christoffel symbols of the Finsler space $\overline{F}^n$ are given by

$$\overline{\gamma}^i_{jk} = \frac{1}{2}\overline{g}^{ir}(\partial_j\overline{g}_{kr} + \partial_k\overline{g}_{rj} - \partial_r\overline{g}_{jk}).$$

In view of Lemma 1.4 and the above expression, we get

Proposition 1.9. *Under a generalized β-conformal change, the Christoffel symbols γ^i_{jk} transform as follows:*

$$\begin{aligned}
\overline{\gamma}^i_{jk} &= \gamma^i_{jk} + \overline{g}^{ir}\left[F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r - \Theta_{(j,k,r)}\left\{B_{jk}b_{0|r} + V_{jkt}N^t_r + (1/2)K_{jk}\sigma_r\right\}\right] \\
&\quad +(g^{im} - e^{\sigma}p\overline{g}^{im})\Theta_{(j,k,m)}\{C_{jkr}N^r_m\},
\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}
2E_{ij} &= b_{i|j} + b_{j|i},\ \ 2F_{ij} = b_{i|j} - b_{j|i}, \\
2B_{ij} &= e^{\sigma}p_{-1}h_{ij} + p_{02}m_im_j, \\
Q_i &= e^{\sigma}p_{-1}y_i + p_0b_i, \\
K_{ij} &= A_1g_{ij} + A_2b_ib_j + A_3(b_iy_j + b_jy_i) + A_4y_iy_j, \\
A_1 &= e^{\sigma}(2p - \beta p_{-1}), A_2 = -\beta p_{02}, A_3 = e^{\sigma}p_{-1} + (\beta^2/L^2)p_{02}\,,\ \ A_4 = e^{\sigma}p_{-2} - (\beta^3/L^4)p_{02},
\end{aligned}$$

Proof. After long but easy calculations, using Lemma 1.4, one can show that

$$\begin{aligned}
\partial_j\overline{g}_{kr} &= e^{\sigma}p\,\partial_jg_{kr} + 2N^l_jV_{krl} + 2B_{kr}b_{0|j} + Q_rb_{k|j} + Q_kb_{r|j} + p_0(b_rb_l\Gamma^l_{kj} + b_lb_k\Gamma^l_{rj}) \\
&\quad + e^{\sigma}p_{-1}((b_ry_l + b_ly_r)\Gamma^l_{kj} + (b_ly_k + b_ky_l)\Gamma^l_{rj}) + e^{\sigma}p_{-2}(y_ry_l\Gamma^l_{kj} + y_ky_l\Gamma^l_{rj}) + K_{kr}\sigma_j.
\end{aligned}$$

The result follows from the above formula and the definition of $\overline{\gamma}^i_{jk}$. □

Propositions 1.5, 1.6, 1.8 and 1.9 constitute the main elementary entities, or building blocks, of the geometry of the transformed space $\overline{F}^n$. As a result, we are now in a position to construct the fundamental geometric objects of such geometry.

Firstly, the following result determines the change of the canonical spray and Cartan nonlinear connection under a generalized β-conformal change.

Theorem 1.10. *Under a generalized β-conformal change, we have:*

(a) *The change of the canonical spray G^i is given by*

$$\overline{G}^i = G^i + D^i,$$

where

$$\begin{aligned} D^i &= \frac{\sigma_0}{2p}\{[2p - \beta p_{-1} - e^\sigma p^2 L^2 s_{-2} - p s_{-1}(2e^\sigma p\beta + e^\sigma p_{-1}L^2 m^2)]y^i - 2e^\sigma p^2 \beta s_0 b^i\} \\ &\quad + \frac{q}{p}e^{-\sigma}F_0^i - \frac{1}{2}L^2\sigma^i + \frac{1}{2}(e^\sigma p E_{00} - 2qF_{\beta 0} + e^\sigma p L^2 \sigma_\beta)(s_0 b^i + s_{-1}y^i); \qquad (1.5) \\ F_0^i &= F_{jk}y^k g^{ij},\ F_{\beta 0} = F_{r0}b^r,\ \sigma_\beta = \sigma_r b^r. \end{aligned}$$

(b) *The change of the Cartan nonlinear connection N_j^i is given by*

$$\overline{N}_j^i = N_j^i + D_j^i,$$

where

$$\begin{aligned} D_j^i &= \frac{e^{-\sigma}}{p}A_j^i - (s_0 b^i + s_{-1}y^i)A_{rj}b^r \\ &\quad -(q b_{0|j} + e^\sigma p L^2 \sigma_j)(s_{-1}b^i + s_{-2}y^i); \qquad (1.6) \\ A_{ij} &:= E_{00}B_{ij} + F_{i0}Q_j + qF_{ij} + E_{j0}Q_i - 2(e^\sigma p\, C_{sij} + V_{sij})D^s \\ &\quad + \frac{1}{2}\sigma_0[2e^\sigma p g_{ij} + 2e^\sigma p_{-1}m_j y_i - 2\beta B_{ij} + e^\sigma p_{-1}(b_i y_j - b_j y_i)] \\ &\quad - \frac{1}{2}\sigma_i(e^\sigma L^2 p_{-1}m_j + 2e^\sigma p y_j) + \frac{1}{2}\sigma_j(2e^\sigma p y_i + e^\sigma L^2 p_{-1}m_i), \\ A_j^i &= g^{li}A_{lj}. \end{aligned}$$

Proof.

(a) Using proposition 1.9 and the expression $\overline{G}^i = \frac{1}{2}\overline{\gamma}_{jk}^i y^j y^k$, we get

$$\begin{aligned} \overline{G}^i &= G^i + \frac{1}{2}\overline{g}^{ir}[2qF_{r0} + E_{00}Q_r - e^\sigma p L^2\sigma_r + \sigma_0(2e^\sigma p y_r + e^\sigma L^2 p_{-1}m_r)] \qquad (1.7) \\ &= G^i + \frac{q}{p}e^{-\sigma}F_0^i - \frac{1}{2}L^2\sigma^i + \frac{1}{2}(e^\sigma p E_{00} - 2qF_{\beta 0} + e^\sigma p L^2\sigma_\beta)(s_0 b^i + s_{-1}y^i) \\ &\quad - \frac{\sigma_0}{2p}\{2e^\sigma p^2\beta s_0 b^i - [2p - p_{-1}\beta - e^\sigma p^2 L^2 s_{-2} - p s_{-1}(2e^\sigma p\beta + e^\sigma p_{-1}L^2 m^2)]y^i\}. \end{aligned}$$

(b) Differentiating D^i with respect to y^j, we have

$$\begin{aligned} D_j^i &:= \dot{\partial}_j D^i = \frac{1}{2}\dot{\partial}_j\{\overline{g}^{ir}[2qF_{r0} + E_{00}Q_r - e^\sigma p L^2\sigma_r + \sigma_0(2e^\sigma p y_r + e^\sigma L^2 p_{-1}m_r)]\} \\ &= \overline{g}^{ir}\{E_{00}B_{rj} + F_{r0}Q_j + qF_{rj} + E_{j0}Q_r - 2(e^\sigma p\, C_{srj} + V_{srj})D^s \\ &\quad + \frac{1}{2}\sigma_0(2e^\sigma p g_{jr} + 2e^\sigma p_{-1}m_j y_r - 2\beta B_{jr} + e^\sigma p_{-1}(b_r y_j - b_j y_r)) \\ &\quad - \frac{1}{2}\sigma_r(e^\sigma L^2 p_{-1}m_j + 2e^\sigma p L l_j) + \frac{1}{2}\sigma_j(2e^\sigma p y_r + e^\sigma L^2 p_{-1}m_r)\} \\ &= \frac{e^{-\sigma}}{p}A_j^i - (s_0 b^i + s_{-1}y^i)A_{rj}b^r - (q b_{0|j} + e^\sigma p L^2\sigma_j)(s_{-1}b^i + s_{-2}y^i). \end{aligned}$$

This ends the proof. □

As a direct consequence of the above theorem, the coefficients of the Berwald connection $B\overline{\Gamma}$ of the transformed Finsler space $\overline{F}^n$ can be computed as follows.

Theorem 1.11. *Under a generalized β-conformal change, the coefficients of the Berwald connection $\overline{G}^i_{jk}$ are given by*

$$\overline{G}^i_{jk} = G^i_{jk} + B^i_{jk},$$

where $B^i_{jk} := \dot{\partial}_k D^i_j$.

Now, we are in a position to announce one of the main results of the present paper. Namely,

Theorem 1.12. *Under a generalized β-conformal change, the coefficients of the Cartan connection $\overline{\Gamma}^i_{jk}$ are given by*

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + D^i_{jk},$$

where

$$\begin{aligned} D^i_{jk} := {} & [(e^{-\sigma}/p)g^{ir} - (s_0 b^i + s_{-1}y^i)b^r - (s_{-1}b^i + s_{-2}y^i)y^r][F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r \\ & + \frac{1}{2}\Theta_{(j,k,r)}\{2e^{\sigma}p\, C_{jkm}D^m_r + 2V_{jkm}D^m_r - K_{jk}\sigma_r - 2B_{jk}b_{0|r}\}]. \end{aligned} \tag{1.8}$$

Proof. To compute $\overline{\Gamma}^i_{jk}$, we use Propositions 1.6, 1.8, 1.9 and Theorem 1.10:

$$\begin{aligned} \overline{\Gamma}^i_{jk} &= \overline{\gamma}^i_{jk} + \overline{g}^{im}(\overline{C}_{jkr}\overline{N}^r_m - \overline{C}_{mkr}\overline{N}^r_j - \overline{C}_{jmr}\overline{N}^r_k) \\ &= \gamma^i_{jk} + (g^{im} - e^{\sigma}p\overline{g}^{im})(C_{jkr}N^r_m - C_{krm}N^r_j - C_{jrm}N^r_k) + \overline{g}^{ir}(B_{jr}b_{0|k} \\ &\quad + B_{kr}b_{0|j} - B_{jk}b_{0|r} + F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r + N^t_j V_{krt} + N^t_k V_{jrt} - N^t_r V_{jkt}) \\ &\quad + \frac{1}{2}\overline{g}^{ir}(\sigma_k K_{jr} + \sigma_j K_{kr} - \sigma_r K_{jk}) + \overline{g}^{im}\{(e^{\sigma}p\, C_{jkr} + V_{jkr})(N^r_m + D^r_m) \\ &\quad - (e^{\sigma}p\, C_{krm} + V_{krm})(N^r_j + D^r_j) - (e^{\sigma}p\, C_{jmr} + V_{jmr})(N^r_k + D^r_k)\} \\ &= \gamma^i_{jk} + g^{im}(C_{jkr}N^r_m - C_{krm}N^r_j - C_{jrm}N^r_k) + \overline{g}^{ir}\{B_{jr}b_{0|k} + B_{kr}b_{0|j} - B_{jk}b_{0|r} \\ &\quad + F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r + \frac{1}{2}(\sigma_k K_{jr} + \sigma_j K_{kr} - \sigma_r K_{jk}) \\ &\quad + e^{\sigma}p\, C_{jkm}D^m_r + V_{jkm}D^m_r - e^{\sigma}p\, C_{rkm}D^m_j - V_{rkm}D^m_j - e^{\sigma}p\, C_{rjm}D^m_k - V_{rjm}D^m_k\} \\ &= \Gamma^i_{jk} + [(e^{-\sigma}/p)g^{ir} - (s_0 b^i + s_{-1}y^i)b^r - (s_{-1}b^i + s_{-2}y^i)y^r][F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r \\ &\quad + \frac{1}{2}\Theta_{(j,k,r)}\{2e^{\sigma}p\, C_{jkm}D^m_r + 2V_{jkm}D^m_r - K_{jk}\sigma_r - 2B_{jk}b_{0|r}\}]. \end{aligned}$$

This completes the proof. $\square$

Corollary 1.13. *The tensor D^i_{jk} has the properties:*

$$D^i_{j0} = B^i_{j0} = D^i_j, \qquad D^i_{00} = 2D^i. \tag{1.9}$$

In what follows we say that A_i, for example, is Cartan-parallel to mean that A_i is parallel with respect to the *horizontal* covariant derivative of Cartan connection: $A_{i|j} = 0$. Similarly, for the other connections existing in the space.

Theorem 1.14. *Under a generalized β-conformal change $L \to \overline{L} = f(e^\sigma L, \beta)$, consider the following two assertions:*

(i) *The covariant vector b_i is Cartan-parallel.*

(ii) *The difference tensor D^i_{jk} vanishes identically.*

Then, we have:

(a) *If **(i)** and **(ii)** hold, then σ is homothetic.*

(b) *If σ is homothetic, then **(i)** and **(ii)** are equivalent.*

Proof.
(a) If $D^i_{jk} = 0$, then, by (1.9), $D^i = 0$ (i.e, $\overline{G}^i = G^i$). Moreover, $b_{j|k} = 0$ implies that $F_{jk} = E_{jk} = 0$. Consequently, (1.7) reduces to

$$pL^2\sigma_r - \sigma_0(2py_r + L^2 p_{-1} m_r) = 0.$$

Now, transvecting with y^r, we get $\sigma_0 = 0$. From which, the above equation implies that $\sigma_r = 0$. That is, σ is homothetic.

(b) Let σ be homothetic and $b_{j|k} = 0$. Then, $D^i = 0$, by (1.5). Consequently, $D^i_{jk} = 0$ by (1.8).

On the other hand, let σ be homothetic and $D^i_{jk} = 0$. Then, by (1.9), $D^i = 0$. Hence, (1.5) reduces to

$$\frac{e^{-\sigma} q}{p} F^i_0 + \frac{1}{2}(e^\sigma p E_{00} - 2qF_{\beta 0})(s_0 b^i + s_{-1} y^i) = 0. \tag{1.10}$$

Transvecting (1.10) with y_i and since $s_0\beta + s_{-1}L^2 \neq 0$ (by Remark 1.7), we get

$$e^\sigma p E_{00} - 2qF_{\beta 0} = 0. \tag{1.11}$$

This, together with (1.10), imply that $F^i_0 = 0$. Consequently, $E_{00} = 0$, by (1.11). Since $F_{ij} = \dot{\partial}_j F_{i0}$ and $E_{j0} = \dot{\partial}_j E_{00}$, then $F_{ij} = 0$ and $E_{j0} = 0$, which leads to $b_{i|0} = b_{0|i} = 0$. Consequently, $0 = D^i_{jk} = \overline{g}^{ir} E_{jk} Q_r$, by (1.8). Hence, $E_{jk} Q_r = 0$ and transvecting this with y^r gives $E_{jk} = 0$. Then, the result follows from the definition of F_{jk} and E_{jk}. □

As a consequence of the above theorem, we have the following interesting special cases.

Theorem 1.15.

(a) *Let the generalized β-conformal change $L \to \overline{L} = f(e^\sigma L, \beta)$ be a conformal change ($\beta = 0$), then D^i_{jk} vanishes identically if and only if σ is homothetic.*

(b) *Let the generalized β-conformal change $L \to \overline{L} = f(e^\sigma L, \beta)$ be a β-change ($\sigma = 0$), then D^i_{ij} vanishes identically if and only if b_i is Cartan-parallel.*

A Finsler space $F^n = (M, L)$ is called a Berwald space if the Berwald connection coefficients G^i_{jk} are function of the positional argument x^i only. As an immediate consequence of Theorems 1.14 and 1.15, we have

Theorem 1.16. *Consider a generalized β-conformal change $L \to \overline{L}$ having the properties that $b_i(x)$ is Cartan-parallel and σ is homothetic. If the original space F^n is a Berwald space, then so is the transformed space $\overline{F}^n$.*

Corollary 1.17. *Let the Finsler structure L on F^n be Riemannian. Assume that $b_i(x)$ is Riemann-parallel and σ is homothetic. Then, the transformed space $\overline{F}^n$ is a Berwald space.*

It is to be noted that Theorem 1.14, Theorem 1.16 and Corollary 1.17 generalize some of Shibata's results [14] and Abed's results [2].

We conclude this section with the following result which determines the coefficients of the fundamental linear connections in Finsler geometry.

Theorem 1.18. *Under the generalized β-conformal change (1.1),*

(a) *the transformed Cartan connection has the form $\overline{C\Gamma} = (\overline{\Gamma}^h_{ij}, \overline{N}^h_i, \overline{C}^h_{ij})$,*

(b) *the transformed Chern connection has the form $\overline{R\Gamma} = (\overline{\Gamma}^h_{ij}, \overline{N}^h_i, 0)$,*

(c) *the transformed Hashiguchi connection has the form $\overline{H\Gamma} = (\overline{G}^h_{ij}, \overline{N}^h_i, \overline{C}^h_{ij})$,*

(d) *the transformed Berwald connection has the form $\overline{B\Gamma} = (\overline{G}^h_{ij}, \overline{N}^h_i, 0)$,*

where the coefficients $\overline{N}^h_i$, $\overline{G}^h_{ij}$ and $\overline{\Gamma}^h_{ij}$ are given by Theorem 1.10, Theorem 1.11 and Theorem 1.12 respectively, whereas the components $\overline{C}^h_{ij}$ are given by Proposition 1.8.

2. Change of the torsion and curvature tensors

In this section, we consider how the torsion and curvature tensors transform under the generalized β-conformal change (1.1).

For an arbitrary Finsler connection $F\Gamma = (\mathbf{F}^i_{jk}, \mathbf{N}^i_j, \mathbf{C}^i_{jk})$ on the space F^n, the (h)h-, (h)hv-, (v)h-, (v)hv- and (v)v-torsion tensors of $F\Gamma$ are respectively given by [5]:

$$\begin{aligned}
\mathbf{T}^i_{jk} &= \mathbf{F}^i_{jk} - \mathbf{F}^i_{kj}, \\
\mathbf{C}^i_{jk} &= \text{the connection parameters } \mathbf{C}^i_{jk}, \\
\mathbf{R}^i_{jk} &= \delta_k \mathbf{N}^i_j - \delta_j \mathbf{N}^i_k, \\
\mathbf{P}^i_{jk} &= \dot{\partial}_k \mathbf{N}^i_j - \mathbf{F}^i_{jk}, \\
\mathbf{S}^i_{jk} &= \mathbf{C}^i_{jk} - \mathbf{C}^i_{kj}.
\end{aligned}$$

The h-, hv- and v-curvature tensors of $F\Gamma$ are respectively given by [5]:

$$
\begin{aligned}
\mathbf{R}^i_{hjk} &= \mathfrak{A}_{(j,k)}\{\delta_k \mathbf{F}^i_{hj} + \mathbf{F}^m_{hj}\mathbf{F}^i_{mk}\} + \mathbf{C}^i_{hm}\mathbf{R}^m_{jk}, \\
\mathbf{P}^i_{hjk} &= \dot{\partial}_k \mathbf{F}^i_{hj} - \mathbf{C}^i_{hk|j} + \mathbf{C}^i_{hm}\mathbf{P}^m_{jk}, \\
\mathbf{S}^i_{hjk} &= \mathfrak{A}_{(j,k)}\{\dot{\partial}_k \mathbf{C}^i_{hj} + \mathbf{C}^m_{hk}\mathbf{C}^i_{mj}\}.
\end{aligned}
$$

The next table provides a comparison concerning the four fundamental linear connections and their associated torsion and curvature tensors. The explicit expressions of such tensors, under a generalized β-conformal change, will be given just after the table. It should be noted that the geometric objects associated with Chern connection, Hashiguchi connection and Berwald connection will be marked by $\star$, $*$ and $\circ$ respectively. For Cartan connection no special symbol is assigned.

Table 1: Fundamental linear connections

		Cartan	**Chern**	**Hashiguchi**	**Berwald**
Connections	$(\mathbf{F}^h_{ij}, \mathbf{N}^h_i, \mathbf{C}^h_{ij})$	$(\Gamma^h_{ij}, N^h_i, C^h_{ij})$	$(\Gamma^h_{ij}, N^h_i, 0)$	$(G^h_{ij}, N^h_i, C^h_{ij})$	$(G^h_{ij}, N^h_i, 0)$
Torsions	**(h)h-tors.** $\mathbf{T}^i_{jk}$	0	0	0	0
	(h)hv-tors. $\mathbf{C}^i_{jk}$	C^i_{jk}	0	C^i_{jk}	0
	(v)h-tors. $\mathbf{R}^i_{jk}$	R^i_{jk}	$\overset{\star}{R}{}^i_{jk} = R^i_{jk}$	$\overset{*}{R}{}^i_{jk} = R^i_{jk}$	$\overset{\circ}{R}{}^i_{jk} = R^i_{jk}$
	(v)hv-tors. $\mathbf{P}^i_{jk}$	$P^i_{jk} = C^i_{jk\|0}$	$\overset{\star}{P}{}^i_{jk} = P^i_{jk}$	0	0
	(v)v-tors. $\mathbf{S}^i_{jk}$	0	0	0	0
Curvatures	**h-curv.** $\mathbf{R}^h_{ijk}$	R^h_{ijk}	$\overset{\star}{R}{}^h_{ijk}$	$\overset{*}{R}{}^h_{ijk}$	$\overset{\circ}{R}{}^h_{ijk}$
	hv-curv. $\mathbf{P}^h_{ijk}$	P^h_{ijk}	$\overset{\star}{P}{}^h_{ijk}$	$\overset{*}{P}{}^h_{ijk}$	$\overset{\circ}{P}{}^h_{ijk}$
	v-curv. $\mathbf{S}^h_{ijk}$	S^h_{ijk}	0	$\overset{*}{S}{}^h_{ijk} = S^h_{ijk}$	0
Covariant derivatives	**h-cov. der.**	$K^i_{j\|k}$	$K^i_{j\overset{\star}{\|}k} = K^i_{j\|k}$	$K^i_{j\overset{*}{\|}k}$	$K^i_{j\overset{\circ}{\|}k} = K^i_{j\overset{*}{\|}k}$
	v-cov. der.	$K^i_j\|_k$	$K^i_j\overset{\star}{\|}_k = \dot{\partial}_k K^i_j$	$K^i_j\overset{*}{\|}_k = K^i_j\|_k$	$K^i_j\overset{\circ}{\|}_k = \dot{\partial}_k K^i_j$

It is to be noted that the explicit expressions of the geometric objects of the above table can be found in [5].

Now, by Theorem 1.18, one can prove the following

Theorem 2.1. *Under a generalized β-conformal change, the torsion and curvature tensors of Cartan, Chern, Hashiguchi and Berwald connections, are given by:*

(a) *For Cartan connection, we have*

$$\begin{aligned}
\overline{C}^i_{jk} &= C^i_{jk} + M^i_{jk},\\
\overline{R}^i_{jk} &= R^i_{jk} + \mathfrak{A}_{(j,k)}\{D^i_{j|k} - (B^i_{jr} + P^i_{jr})D^r_k\},\\
\overline{P}^i_{jk} &= P^i_{jk} - D^i_{jk} + B^i_{jk},\\
\overline{R}^i_{hjk} &= R^i_{hjk} + 2S^i_{hrt}D^r_j D^t_k + M^i_{ht}R^t_{jk} - \mathfrak{A}_{(j,k)}\{A^i_{hj|k} - A^i_{hjt}D^t_k\\
&\quad + D^i_{tj}D^t_{hk} + P^i_{hjt}D^t_k + M^i_{rh}P^r_{jt}D^t_k - M^i_{th}D^t_{j|k} + M^i_{ht}B^t_{jr}D^r_k\},\\
\overline{P}^i_{hjk} &= P^i_{hjk} - 2S^i_{thk}D^t_j - A^i_{hjk} + C^i_{tk}D^t_{hj} - C^t_{hk}D^i_{tj} + M^i_{th}P^t_{jk} + A^i_{jt}M^t_{hk}\\
&\quad - M^i_{tk}A^t_{hj} - M^i_{hk|j} + M^i_{th}B^t_{jk} + M^i_{tkh}D^t_j + C^i_{rh}M^r_{kt}D^t_j - M^i_{hr}C^r_{tk}D^t_j,\\
\overline{S}^i_{hjk} &= S^i_{hjk} + \mathfrak{A}_{(j,k)}\{C^t_{hk}M^i_{tj} - C^i_{tk}M^t_{hj} - M^i_{tk}M^t_{hj}\},
\end{aligned}$$

where $A^i_{jk} = -D^i_{jk} - C^i_{jt}D^t_k, \quad A^i_{jkh} = \dot{\partial}_h A^i_{jk}$ *and* $M^i_{jkh} = \dot{\partial}_h M^i_{jk}$.

(b) *For Chern Connection, we have*

$$\begin{aligned}
\overline{\overset{\star}{R}}{}^i_{jk} &= R^i_{jk} + \mathfrak{A}_{(j,k)}\{D^i_{j|k} - (B^i_{jr} + P^i_{jr})D^r_k\},\\
\overline{\overset{\star}{P}}{}^i_{jk} &= P^i_{jk} - D^i_{jk} + B^i_{jk},\\
\overline{\overset{\star}{R}}{}^i_{hjk} &= \overset{\star}{R}{}^i_{hjk} + \mathfrak{A}_{(j,k)}\{D^i_{hj|k} - D^i_{hjt}D^t_k - D^i_{tj}D^t_{hk} - \overset{\star}{P}{}^i_{hjt}D^t_k\},\\
\overline{\overset{\star}{P}}{}^i_{hjk} &= \overset{\star}{P}{}^i_{hjk} + D^i_{hjk},
\end{aligned}$$

where $D^i_{jkh} = \dot{\partial}_h D^i_{jk}$.

(c) *For Hashiguchi connection, we have*

$$\begin{aligned}
\overline{\overset{*}{C}}{}^i_{jk} &= C^i_{jk} + M^i_{jk},\\
\overline{\overset{*}{R}}{}^i_{jk} &= R^i_{jk} + \mathfrak{A}_{(j,k)}\{D^i_{\underset{*}{j|k}} - B^i_{jr}D^r_k\},\\
\overline{\overset{*}{R}}{}^i_{hjk} &= \overset{*}{R}{}^i_{hjk} + 2\overset{*}{S}{}^i_{hrt}D^r_j D^t_k + M^i_{ht}\overset{*}{R}{}^t_{jk} - \mathfrak{A}_{(j,k)}\{H^i_{\underset{*}{hj|k}} - (\dot{\partial}_t H^i_{hj})D^t_k\\
&\quad + D^i_{tj}D^t_{hk} + \overset{*}{P}{}^i_{hjt}D^t_k - M^i_{th}D^t_{\underset{*}{j|k}} + M^i_{ht}B^t_{jr}D^r_k\},\\
\overline{\overset{*}{P}}{}^i_{hjk} &= \overset{*}{P}{}^i_{hjk} - 2\overset{*}{S}{}^i_{thk}D^t_j - \dot{\partial}_k H^i_{hj} + C^i_{tk}D^t_{hj} - C^t_{hk}D^i_{tj} + H^i_{jt}M^t_{hk} - M^i_{tk}H^t_{hj} - M^i_{\underset{*}{hk|j}}\\
&\quad + M^i_{th}B^t_{jk} + M^i_{kth}D^t_j + C^i_{rh}M^r_{kt}D^t_j - M^i_{hr}C^r_{tk}D^t_j\\
\overline{\overset{*}{S}}{}^i_{hjk} &= S^i_{hjk} + \mathfrak{A}_{(j,k)}\{C^t_{hk}M^i_{tj} - C^i_{tk}M^t_{hj} - M^i_{tk}M^t_{hj}\},
\end{aligned}$$

where $H^i_{jk} = -B^i_{jk} - C^i_{jt}D^t_k$.

(d) *For Berwald connection, we have*

$$\overline{\overset{\circ}{R}}{}^i_{jk} = R^i_{jk} + \mathfrak{A}_{(j,k)}\{D^i_{\underset{\circ}{j|k}} - B^i_{jr}D^r_k\},$$

$$\overline{\overset{\circ}{R}}{}^i_{hjk} = \overset{\circ}{R}{}^i_{hjk} - \mathfrak{A}_{(j,k)}\{B^i_{\underset{\circ}{hj|k}} - B^i_{hjt}D^t_k + D^i_{tj}D^t_{hk} + \overset{\circ}{P}{}^i_{hjt}D^t_k\},$$

$$\overline{\overset{\circ}{P}}{}^i_{hjk} = \overset{\circ}{P}{}^i_{hjk} + B^i_{hjk},$$

where $B^i_{jkh} = \dot{\partial}_h B^i_{jk}$.

Corollary 2.2. *Under a generalized β-conformal change, for which the covariant vector field b_i is Cartan-parallel and σ is homothetic, we have:*

(a) *The torsion and curvature tensors of Chern connection are invariant.*

(b) *The torsion and curvature tensors of Berwald connection are invariant.*

(c) *For Cartan connection, R^i_{jk} and P^i_{jk} are invariant and*

$$\overline{C}{}^i_{jk} = C^i_{jk} + M^i_{jk},$$
$$\overline{R}{}^i_{hjk} = R^i_{hjk} + M^i_{ht}R^t_{jk},$$
$$\overline{P}{}^i_{hjk} = P^i_{hjk} + M^i_{th}P^t_{jk} - M^i_{hk|j},$$
$$\overline{S}{}^i_{hjk} = S^i_{hjk} + \mathfrak{A}_{(j,k)}\{C^t_{hk}M^i_{tj} - C^i_{tk}M^t_{hj} - M^i_{tk}M^t_{hj}\}.$$

(d) *For Hashiguchi connection, $\overset{*}{R}{}^i_{jk}$ is invariant and*

$$\overline{\overset{*}{C}}{}^i_{jk} = C^i_{jk} + M^i_{jk},$$
$$\overline{\overset{*}{R}}{}^i_{hjk} = \overset{*}{R}{}^i_{hjk} + M^i_{ht}\overset{*}{R}{}^t_{jk},$$
$$\overline{\overset{*}{P}}{}^i_{hjk} = \overset{*}{P}{}^i_{hjk} - M^i_{\underset{*}{hk|j}},$$
$$\overline{\overset{*}{S}}{}^i_{hjk} = \overset{*}{S}{}^i_{hjk} + \mathfrak{A}_{(j,k)}\{C^t_{hk}M^i_{tj} - C^i_{tk}M^t_{hj} - M^i_{tk}M^t_{hj}\}.$$

A Finsler space F^n is called a Landesberg space if the hv-curvature tensor P^i_{hjk} of $C\Gamma$ vanishes, or equivalently $P^i_{jk} = 0$.

By Corollary 2.2, P^i_{jk} is invariant under a generalized β-conformal change for which b_i is Cartan-parallel and σ is homothetic. Hence, we have the following

Theorem 2.3. *A Landesberg space remains Landesberg under a generalized β-conformal change if b_i is Cartan-parallel and σ is homothetic.*

By the above theorem and the fact that the hv-curvature tensor P^i_{hjk} of a Riemannian space vanishes identically, we have

Corollary 2.4. *Under a generalized β-conformal change, a Riemannian space F^n is transformed to a Landesberg space $\overline{F}^n$ if b_i is Riemann-parallel and σ is homothetic.*

A Finsler space F^n is called locally Minkowskian if F^n is a Berwald space and the h-curvature tensor R^i_{hjk} vanishes.

Theorem 2.5. *Assume that the covariant vector $b_i(x)$ is Cartan-parallel and σ is homothetic. If F^n is locally Minkowskian, then so is the space $\overline{F}^n$.*

Proof.

We prove firstly that if the covariant vector $b_i(x)$ is Cartan-parallel and the change is homothetic, then $\overline{R}^i_{hjk}$ vanishes if and only if R^i_{hjk} vanishes. By Corollary 2.2, $\overline{R}^i_{hjk} = R^i_{hjk} + M^i_{hm} R^m_{jk}$. If $R^i_{hjk} = 0$, then R^i_{jk}=0 and hence $\overline{R}^i_{hjk} = 0$. Conversely, if $\overline{R}^i_{hjk} = 0$, then $R^i_{hjk} + M^i_{ht} R^t_{jk} = 0$. By transvection with y^h, we obtain $R^i_{jk} = 0$ and hence $R^i_{hjk} = 0$.

Now, the result follows from the above fact and Theorem 1.16. □

Theorem 2.6. *Under a generalized β-conformal change, a Riemannian space F^n is transformed to a locally Minkowskian space $\overline{F}^n$ if b_i is Riemann-parallel, σ is homothetic and R^i_{hjk} vanishes.*

Proof.

Follows directly from Corollary 1.17. □

It is to be noted that Theorem 2.3, Corollary 2.4, Theorem 2.5 and Theorem 2.6 generalize various results of Shibata [14].

3. Concluding remarks

In this paper, we have introduced a generalized change, which combines both β-change and conformal change in a general setting. We have refered to this change as a generalized β-conformal change. Many of the known Finsler changes in the literatures may be obtained from the generalized β-conformal change as special cases.

We will mention some interesting special cases. In these special cases, we restrict ourselves to the difference tensor D^i_{jk} only.

• When the generalized β-conformal change (1.1) is a β-change: $\overline{L} = f(L, \beta)$, the difference tensor (1.8) takes the form:

$$\begin{aligned} D^i_{jk} = &\{(1/p)g^{ir} - (s_0 b^i + s_{-1} y^i) b^r - (s_{-1} b^i + s_{-2} y^i) y^r\}\{B_{jr} b_{0|k} + B_{kr} b_{0|j} \\ &- B_{jk} b_{0|r} + F_{rk} Q_j + F_{rj} Q_k + E_{jk} Q_r + p\, C_{jkm} D^m_r + V_{jkm} D^m_r \\ &- p\, C_{rkm} D^m_j - V_{rkm} D^m_j - p\, C_{rjm} D^m_k - V_{rjm} D^m_k\}. \end{aligned}$$

This is the case studied by Shibata [14].

• When the generalized β-conformal change (1.1) is a β-conformal change: $\overline{L} = e^\sigma L + \beta$, the difference tensor (1.8) takes the form:

$$\begin{aligned} D^i_{jk} = &[\tau^{-1} g^{ir} - (1/\overline{L}\tau)(y^i b^r + y^r b^i) + \mu l^i l^r][F_{rk} Q_j + F_{rj} Q_k + E_{jk} Q_r \\ &+ (1/2)\Theta_{(j,k,r)}\{2\tau C_{jkm} D^m_r + 2 V_{jkm} D^m_r - K_{jk}\sigma_r - (e^\sigma/L) h_{jk} b_{0|r}\}], \end{aligned}$$

where $\tau := e^\sigma \frac{\overline{L}}{L}$ and $\mu := \frac{L}{\tau \overline{L}^2}(Lb^2 + \beta e^\sigma)$.
This is the case studied by Abed [2].

• When the generalized β-conformal change (1.1) is a generalized Randers change: $\overline{L} = L + \beta$, with L Finslerian, the difference tensor (1.8) takes the form:

$$\begin{aligned} D^i_{jk} = {} & [\tau^{-1} g^{ir} - \frac{1}{\overline{L}\tau}(y^i b^r + y^r b^i) + \mu l^i l^r][F_{rk}Q_j + F_{rj}Q_k + E_{jk}Q_r \\ & +(1/2)\Theta_{(j,k,r)}\{2\tau C_{jkm}D^m_r + 2V_{jkm}D^m_r - (1/L)h_{jk}b_{0|r}\}], \end{aligned}$$

This is the case studied by Matsumoto [10], Tamim and Youssef [16] and others.

• When the generalized β-conformal change (1.1) is a Kropina change: $\overline{L} = L^2/\beta$, with L Riemannaian, the difference tensor (1.8) takes the form:

$$\begin{aligned} D^i_{jk} = {} & \frac{1}{2L^4 b^2 \beta^3}\left[L^2 b^2 g^{ir} - (L^2 b^i - 2\beta y^i)b^r + 2(m^2 y^i + \beta m^i)y^r\right] \\ & .[\beta L^2\left(F_{rk}(3L^2 b_j - 4\beta y_j) + F_{rj}(3L^2 b_k - 4\beta y_k) + E_{jk}(3L^2 b_r - 4\beta y_r)\right) \\ & +(1/2)\Theta_{(j,k,r)}\{2\beta^5 V_{jkm}D^m_r + 4L^2(\beta^2 h_{jk} + 3L^2 m_j m_k)b_{0|r}\}]. \end{aligned}$$

This is the case studied by Kropina [8], Matsumoto [9], Shibata [13] and others.

• When the generalized β-conformal change (1.1) is a conformal change: $\overline{L} = e^\sigma L$, the difference tensor (1.8) takes the form:

$$\begin{aligned} D^i_{jk} = {} & \sigma_j \delta^i_k + \sigma_k \delta^i_j - \sigma^i g_{jk} + y_j C^i_{km}\sigma^m + y_k C^i_{jm}\sigma^m - y^i C_{jkm}\sigma^m - \sigma_0 C^i_{jk} \\ & +L^2(C_{jkm}C^{mi}_r\sigma^r - C^i_{km}C^m_{jr}\sigma^r - C^i_{jm}C^m_{kr}\sigma^r). \end{aligned}$$

This is the case studied by Hashiguchi [5], Izumi [7], Youssef et al. [18] and others.

• When the generalized β-conformal change (1.1) is a C-conformal (resp. h-conformal) change: $\overline{L} = e^\sigma L$, with σ enjoying the property that $C^i_{jk}\sigma_i = 0$ (resp. $C^i_{jk}\sigma_i = \frac{1}{n-1}C^i\sigma_i h_{jk}$), the difference tensor (1.8) takes the form:

$$D^i_{jk} = \sigma_j \delta^i_k + \sigma_k \delta^i_j - \sigma^i g_{jk} - \sigma_0 C^i_{jk}.$$

$$(\textit{resp.}\quad D^i_{jk} = \sigma_j \delta^i_k + \sigma_k \delta^i_j - \sigma^i g_{jk} - \sigma_0 C^i_{jk} + \frac{1}{n-1}C^r\sigma_r(y_j h^i_k + y_k h^i_j - y^i h_{jk} - L^2 C^i_{jk})).$$

This is the case studied by Hashiguchi [5] (resp. Izumi [7]).

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